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Diagonal Projection of a Double Dispersion Lorentzian Function

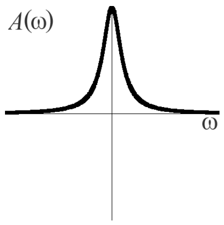
A Mathematical Paradox Concerning the Integrated Volume of a Two-Dimensional Lorentzian

This problem concerns a two-dimensional "Double Dispersion" Lorentzian function. We will show that the analytical calculation of the volume bounded by this function that we know intuitively, and by symmetry, to have zero net volume leads to a paradoxical result.

To begin with, we define the basic Lorentzian functions and determine their properties. The two Lorentzian functions we investigate are known as the **absorption** and **dispersion** functions. These functions occur frequently in the physical sciences.

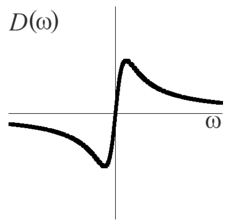
The one-dimensional Lorentzian absorption and dispersion functions are defined as:

$$A(\omega) = \frac{1}{(1+\omega^2)}$$



and

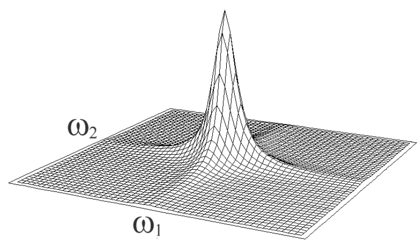
$$D(\omega) = \frac{\omega}{(1+\omega^2)}$$

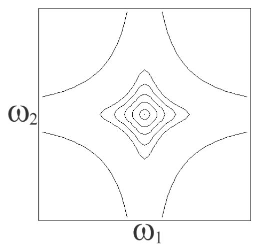


$A(\omega)$ converges to zero as $1/\omega^2$ and the area under its curve is equal to π. But $D(\omega)$ converges to zero much more slowly as $1/\omega$ and, by symmetry, the area under its curve is equal to zero, the positive area for $\omega>0$ being cancelled by the negative area for $\omega<0$. These two functions are related by Hilbert transformation or, more graphically, by the fact that a plot of $D(\omega)$ vs. $A(\omega)$ yields a circle.

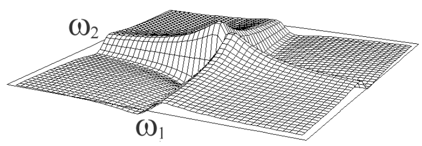
The two-dimensional versions of these functions are the double absorption and double dispersion Lorentzian functions $A(\omega_1, \omega_2) = A(\omega_1) \cdot A(\omega_2)$ and $D(\omega_1, \omega_2) = -D(\omega_1) \cdot D(\omega_2)$, as follows:

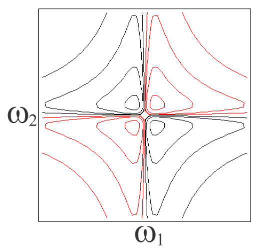
$$A(\omega_1, \omega_2) = \frac{1}{(1+\omega_1^2)(1+\omega_2^2)}$$





$$D(\omega_1, \omega_2) = \frac{-\omega_1\omega_2}{(1+\omega_1^2)(1+\omega_2^2)}$$





To calculate the volumes bounded by these functions, we simply integrate over all space. By symmetry, the order of integration should be irrelevant. For $A(\omega_1, \omega_2)$ the volume is π^2:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega_1, \omega_2)\,d\omega_1 d\omega_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega_1, \omega_2)\,d\omega_2 d\omega_1 = \pi^2$$

For $D(\omega_1, \omega_2)$, we expect this function, intuitively, to have zero net volume when integrated over all space. Hence, we expect that if we compute an arbitrary **projection** (integral in one dimension) of the function and then integrate that projection, the result will be zero. This is indeed the case when we project onto ω_1 or ω_2 and then integrate in the other direction, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\omega_1,\omega_2)d\omega_1 d\omega_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\omega_1,\omega_2)d\omega_2 d\omega_1 = 0$$

Almost trivially, the result for $D(\omega_1, \omega_2)$ is identical in cylindrical polar coordinates:

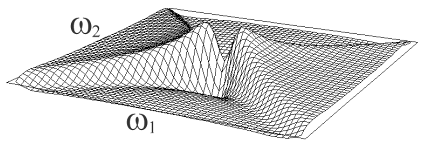
$$\int_0^{2\pi}\int_0^{\infty} D(r,\theta)rdrd\theta = \int_0^{2\pi}\int_0^{\infty} D(r,\theta)\frac{-r^2\cos\theta\sin\theta}{(1+r^2\cos^2\theta)(1+r^2\sin^2\theta)}rdrd\theta = 0$$

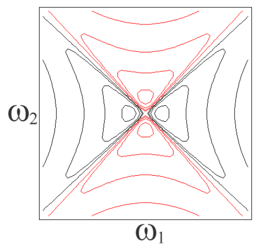
The same does not hold, however, for 45° projections in Cartesian space. Calculation of the diagonal projections of $D(\omega_1, \omega_2)$ is equivalent to calculation of the 90° ω_1 and ω_2 projections of a function generated by rotation of $D(\omega_1,\omega_2)$ by 45° in the $\omega_1\omega_2$ plane. The required (anticlockwise) transformation is:

$$\omega_1 \to \frac{\omega_1 - \omega_2}{\sqrt{2}} \qquad \omega_2 \to \frac{\omega_1 + \omega_2}{\sqrt{2}}$$

This gives:

$$G(\omega_1,\omega_2) = \frac{-\frac{1}{2}(\omega_1-\omega_2)(\omega_1+\omega_2)}{(1+\frac{1}{2}(\omega_1-\omega_2)^2)(1+\frac{1}{2}(\omega_1+\omega_2)^2)}$$

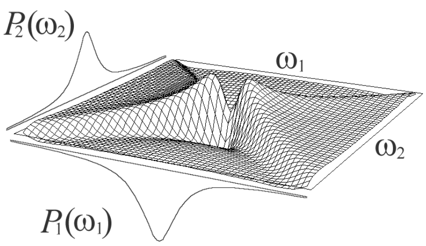


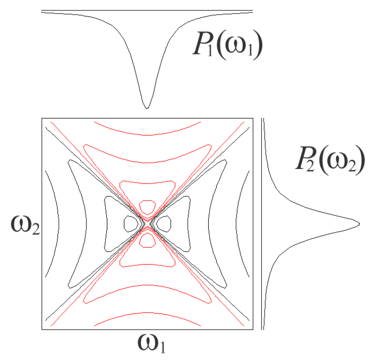


Our problem arises when we come to compute the **projections** of $G(\omega_1, \omega_2)$ onto the ω_1 and ω_2 axes. These are solvable analytically and are given by:

$$\int_{-\infty}^{\infty} G(\omega_1,\omega_2)d\omega_2 = \frac{-\sqrt{2}\pi}{2+\omega_1^2} = P_1(\omega_1) \quad \text{and} \quad \int_{-\infty}^{\infty} G(\omega_1,\omega_2)d\omega_1 = \frac{\sqrt{2}\pi}{2+\omega_2^2} = P_2(\omega_2)$$

Note that $P_1(\omega_1)$ and $P_2(\omega_2)$ are simple one-dimensional Lorentzian absorption functions (and therefore have finite areas) and note also that they are of opposite sign as illustrated below:





If we then calculate the integrals of these two projections, we obtain:

$$\int_{-\infty}^{\infty} P_1(\omega_1)d\omega_1 = -\pi^2 \quad \text{and} \quad \int_{-\infty}^{\infty} P_2(\omega_2)d\omega_2 = \pi^2$$

This is somewhat at odds with our intuitive picture and suggests that the volume bounded by $G(\omega_1, \omega_2)$, rather than being zero, has a magnitude of π^2 and is either positive or negative depending on the order of integration.

By comparison, and as if to add insult to injury, when we calculate the volume integral over all space of $G(\omega_1, \omega_2)$ in cylindrical polar coordinates, the result is zero:

$$\int_0^{2\pi} \int_0^{\infty} G(r,\theta) r dr d\theta = \int_0^{2\pi} \int_0^{\infty} D(r,\theta + \tfrac{\pi}{4}) r dr d\theta = 0$$

Given that π^2 and $-\pi^2$ are not the same as zero, what is wrong with the calculation of the Cartesian projections of $G(\omega_1, \omega_2)$?

Please send comments and responses to ***michael.woodley@quincunx.com***.